

CAPITULATION IN THE ABSOLUTELY ABELIAN EXTENSIONS OF SOME FIELDS $\mathbb{Q}(\sqrt{p_1 p_2 q}, \sqrt{-1})$

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ABSTRACT. We study the capitulation of 2-ideal classes of an infinite family of imaginary bicyclic biquadratic number fields consisting of fields $\mathbb{k} = \mathbb{Q}(\sqrt{p_1 p_2 q}, i)$, where $i = \sqrt{-1}$ and $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod{4}$ are different primes. For each of the three quadratic extensions \mathbb{K}/\mathbb{k} inside the absolute genus field $\mathbb{k}^{(*)}$ of \mathbb{k} , we compute the capitulation kernel of \mathbb{K}/\mathbb{k} . Then we deduce that each strongly ambiguous class of $\mathbb{k}/\mathbb{Q}(i)$ capitulates already in $\mathbb{k}^{(*)}$, which is smaller than the relative genus field $(\mathbb{k}/\mathbb{Q}(i))^*$.

1. INTRODUCTION AND NOTATIONS

Let k be an algebraic number field and let $\mathbf{Cl}_2(k)$ denote its 2-class group, that is the 2-Sylow subgroup of the ideal class group, $\mathbf{Cl}(k)$, of k . We denote by $k^{(*)}$ the absolute genus field of k . Suppose F is a finite extension of k , then we say that an ideal class of k capitulates in F if it is in the kernel of the homomorphism

$$J_F : \mathbf{Cl}(k) \longrightarrow \mathbf{Cl}(F)$$

induced by extension of ideals from k to F . An important problem in Number Theory is to determine explicitly the kernel of J_F , which is usually called the capitulation kernel. If F is the relative genus field of a cyclic extension K/k , which we denote by $(K/k)^*$ and that is the maximal unramified extension of K which is obtained by composing K and an abelian extension over k , F. Terada states in [13] that all the ambiguous ideal classes of K/k , which are classes of K fixed under any element of $\text{Gal}(K/k)$, capitulate in $(K/k)^*$. If F is the absolute genus field of an abelian extension K/\mathbb{Q} , then H. Furuya confirms in [14] that every strongly ambiguous class of K/\mathbb{Q} , that is an ambiguous ideal class containing at least one ideal invariant under any element of $\text{Gal}(K/\mathbb{Q})$, capitulates in F . In this paper, we construct a family of number fields k for which all the strongly ambiguous classes of $k/\mathbb{Q}(i)$ capitulate in $k^{(*)} \subsetneq (k/\mathbb{Q}(i))^*$.

Let $\mathbb{k} = \mathbb{Q}(\sqrt{d}, i)$ and \mathbb{K} be an unramified quadratic extension of \mathbb{k} that is abelian over \mathbb{Q} . Denote by $\text{Am}_s(\mathbb{k}/\mathbb{Q}(i))$ the group of the strongly ambiguous classes of $\mathbb{k}/\mathbb{Q}(i)$. In [5], we studied the capitulation problem in the absolutely abelian extensions of \mathbb{k} for $d = 2pq$ and $p \equiv q \equiv 1 \pmod{4}$ are different primes,

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and in [6], we have dealt with the same problem assuming $p \equiv -q \equiv 1 \pmod{4}$. In [7, 8, 9] and under the assumption $\mathbf{Cl}_2(\mathbb{k}) \simeq (2, 2, 2)$, we studied the capitulation problem of the 2-ideal classes of \mathbb{k} in its fourteen unramified extensions, within the first Hilbert 2-class field of \mathbb{k} , and we gave the abelian type invariants of the 2-class groups of these fourteen fields, additionally we determined the structure of the metabelian Galois group $G = \text{Gal}(\mathbb{k}_2^{(2)}/\mathbb{k})$ of the second Hilbert 2-class field $\mathbb{k}_2^{(2)}$ of \mathbb{k} . Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod{4}$ be different primes and $d = p_1 p_2 q$, it is the purpose of the present article to pursue this research project. We will compute the capitulation kernel of \mathbb{K}/\mathbb{k} and we will deduce that $\text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) \subseteq \ker J_{\mathbb{k}^{(*)}}$. As an application we will determine these kernels when $\mathbf{Cl}_2(\mathbb{k})$ is of type $(2, 2, 2)$.

Let k be a number field, during this paper, we adopt the following notations:

- $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod{4}$ are different primes.
- \mathbb{k} : denotes the field $\mathbb{Q}(\sqrt{p_1 p_2 q}, \sqrt{-1})$.
- κ_K : the capitulation kernel of an unramified extension K/\mathbb{k} .
- \mathcal{O}_k : the ring of integers of k .
- E_k : the unit group of \mathcal{O}_k .
- W_k : the group of roots of unity contained in k .
- F.S.U : the fundamental system of units.
- k^+ : the maximal real subfield of k , if it is a CM-field.
- $Q_k = [E_k : W_k E_{k^+}]$ is Hasse's unit index, if k is a CM-field.
- $q(k/\mathbb{Q}) = [E_k : \prod_{i=1}^s E_{k_i}]$ is the unit index of k , if k is multiquadratic, where k_1, \dots, k_s are the quadratic subfields of k .
- $k^{(*)}$: the absolute genus field of k .
- $\mathbf{Cl}_2(k)$: the 2-class group of k .
- $i = \sqrt{-1}$.
- ϵ_m : the fundamental unit of $\mathbb{Q}(\sqrt{m})$, if $m > 1$ is a square-free integer.
- $N(a)$: denotes the absolute norm of a number a i.e. $N_{k/\mathbb{Q}}(a)$, where $k = \mathbb{Q}(\sqrt{a})$.
- $x \pm y$ means $x + y$ or $x - y$ for some numbers x and y .

Our main theorem is.

Theorem. *Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod{4}$ be different primes and put $\mathbb{k} = \mathbb{Q}(\sqrt{p_1 p_2 q}, \sqrt{-1})$. Denote by $\text{Am}_s(\mathbb{k}/\mathbb{Q}(i))$ the group of the strongly ambiguous classes of $\mathbb{k}/\mathbb{Q}(i)$. If \mathbb{K} is an unramified quadratic extension of \mathbb{k} that is abelian over \mathbb{Q} , then*

1. $|\kappa_{\mathbb{K}}| = 2, 4$ or 8 .
2. $\kappa_{\mathbb{K}} \subset \text{Am}_s(\mathbb{k}/\mathbb{Q}(i))$.
3. $\text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) \subseteq \kappa_{\mathbb{k}^{(*)}}$.

2. Preliminary results

Let us first collect some results that will be useful in what follows.

Let k_j , $1 \leq j \leq 3$, be the three real quadratic subfields of a biquadratic bi-cyclic real number field K_0 and $\epsilon_j > 1$ be the fundamental unit of k_j . Since

$\alpha^2 N_{K_0/\mathbf{Q}}(\alpha) = \prod_{j=1}^3 N_{K_0/k_j}(\alpha)$ for any $\alpha \in K_0$, the square of any unit of K_0 is in the group generated by the ϵ_j 's, $1 \leq j \leq 3$. Hence, to determine a fundamental system of units of K_0 it suffices to determine which of the units in $B := \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_1\epsilon_2, \epsilon_1\epsilon_3, \epsilon_2\epsilon_3, \epsilon_1\epsilon_2\epsilon_3\}$ are squares in K_0 (see [16] or [17]). Put $K = K_0(i)$, then to determine a F.S.U of K , we will use the following result (see [2, p.18]) that the first author has deduced from a theorem of Hasse [15, §21, Satz 15].

Lemma 2.1. *Let $n \geq 2$ be an integer and ξ_n a 2^n -th primitive root of unity, then*

$$\xi_n = \frac{1}{2}(\mu_n + \lambda_n i), \quad \text{where} \quad \mu_n = \sqrt{2 + \mu_{n-1}}, \quad \lambda_n = \sqrt{2 - \mu_{n-1}},$$

$$\mu_2 = 0, \lambda_2 = 2 \quad \text{and} \quad \mu_3 = \lambda_3 = \sqrt{2}.$$

Let n_0 be the greatest integer such that ξ_{n_0} is contained in K , $\{\epsilon'_1, \epsilon'_2, \epsilon'_3\}$ a F.S.U of K_0 and ϵ a unit of K_0 such that $(2 + \mu_{n_0})\epsilon$ is a square in K_0 (if it exists). Then a F.S.U of K is one of the following systems:

1. $\{\epsilon'_1, \epsilon'_2, \epsilon'_3\}$ if ϵ does not exist,
2. $\{\epsilon'_1, \epsilon'_2, \sqrt{\xi_{n_0}\epsilon}\}$ if ϵ exists; in this case $\epsilon = \epsilon'_1{}^{i_1} \epsilon'_2{}^{i_2} \epsilon'_3$, where $i_1, i_2 \in \{0, 1\}$ (up to a permutation).

Lemma 2.2 ([1], Lemma 5). *Let $d > 1$ be a square-free integer and $\epsilon_d = x + y\sqrt{d}$, where x, y are integers or semi-integers. If $N(\epsilon_d) = 1$, then $2(x+1)$, $2(x-1)$, $2d(x+1)$ and $2d(x-1)$ are not squares in \mathbb{Q} .*

Lemma 2.3 ([1], Lemma 6). *Let $q \equiv -1 \pmod{4}$ be a prime and $\epsilon_q = x + y\sqrt{q}$ be the fundamental unit of $\mathbb{Q}(\sqrt{q})$. Then x is an even integer, $x \pm 1$ is a square in \mathbb{N} and $2\epsilon_q$ is a square in $\mathbb{Q}(\sqrt{q})$.*

Lemma 2.4 ([1], Lemma 7). *Let p be an odd prime and $\epsilon_{2p} = x + y\sqrt{2p}$. If $N(\epsilon_{2p}) = 1$, then $x \pm 1$ is a square in \mathbb{N} and $2\epsilon_{2p}$ is a square in $\mathbb{Q}(\sqrt{2p})$.*

Lemma 2.5 ([2], 3.(1) p.19). *Let $d > 2$ be a square-free integer and $k = \mathbb{Q}(\sqrt{d}, i)$, put $\epsilon_d = x + y\sqrt{d}$.*

1. *If $N(\epsilon_d) = -1$, then $\{\epsilon_d\}$ is a F.S.U of k .*
2. *If $N(\epsilon_d) = 1$, then $\{\sqrt{i\epsilon_d}\}$ is a F.S.U of k if and only if $x \pm 1$ is a square in \mathbb{N} i.e. $2\epsilon_d$ is a square in $\mathbb{Q}(\sqrt{d})$. Else $\{\epsilon_d\}$ is a F.S.U of k (this result is also in [18]).*

Lemma 2.6. *Let $d \equiv 1 \pmod{4}$ be a positive square free integer and $\epsilon_d = x + y\sqrt{d}$ be the fundamental unit of $\mathbb{Q}(\sqrt{d})$. Assume $N(\epsilon_d) = 1$, then*

1. *$x+1$ and $x-1$ are not squares in \mathbb{N} i.e. $2\epsilon_d$ is not a square in $\mathbb{Q}(\sqrt{d})$.*
2. *For all prime p dividing d , $p(x+1)$ and $p(x-1)$ are not squares in \mathbb{N} .*

Proof. 1. As $d \equiv 1 \pmod{4}$, then by [3, Corollaire 3.2] the unit index of $\mathbb{Q}(\sqrt{d}, i)$ is equal to 1, hence by [2, Applications (ii)] we get that $2\epsilon_d$ is not a square in $\mathbb{Q}(\sqrt{d})$, this is equivalent to $x+1$ and $x-1$ are not squares in \mathbb{N} .

2. Assume $p(x+1)$ or $p(x-1)$ is a square in \mathbb{N} , then, by the decomposition uniqueness in \mathbb{Z} , there exist y_1, y_2 in \mathbb{Z} such that

$$\begin{cases} x \pm 1 = py_1^2, \\ x \mp 1 = d'y_2^2; \end{cases} \quad \text{and} \quad \begin{cases} y = y_1y_2, \\ d = pd'; \end{cases}$$

thus $p(x \pm 1) = p^2y_1^2$ and $p(x \mp 1) = p^2y_1^2 \mp 2p$. This in turn yields that $p^2(x^2 - 1) = p^2y_1^2(p^2y_1^2 \mp 2p)$; as $x^2 - 1 = y^2d$, so we get $y^2d = y_1^2(p^2y_1^2 \mp 2p)$, and $y_2^2d = p^2y_1^2 \mp 2p$. Since $d \equiv 1 \pmod{4}$ and $p \equiv \pm 1 \pmod{4}$, we deduce that $\mp 2 \equiv y_1^2 - y_2^2 \pmod{4}$. On the other hand, we know that $a^2 \equiv 0$ or $1 \pmod{4}$ for all $a \in \mathbb{Z}$, thus $\mp 2 \equiv 0, 1$ or $-1 \pmod{4}$. Which is absurd. \square

3. F.S.U OF SOME CM-FIELDS

As $\mathbb{k} = \mathbb{Q}(\sqrt{p_1p_2q}, i)$, so \mathbb{k} admits three unramified quadratic extensions that are abelian over \mathbb{Q} , which are $\mathbb{K}_1 = \mathbb{k}(\sqrt{p_1}) = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2q}, i)$, $\mathbb{K}_2 = \mathbb{k}(\sqrt{p_2}) = \mathbb{Q}(\sqrt{p_2}, \sqrt{p_1q}, i)$ and $\mathbb{K}_3 = \mathbb{k}(\sqrt{q}) = \mathbb{Q}(\sqrt{q}, \sqrt{p_1p_2}, i)$. Put $\epsilon_{p_1p_2q} = x + y\sqrt{p_1p_2q}$. In what follows, we determine the F.S.U's of \mathbb{K}_j , $1 \leq j \leq 3$.

3.1. F.S.U of the field \mathbb{K}_1 . Let $\mathbb{K}_1 = \mathbb{k}(\sqrt{p_1}) = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2q}, i)$.

Proposition 3.1. *Keep the previous notations and put $\epsilon_{p_2q} = a + b\sqrt{p_2q}$.*

1. *If $a \pm 1$ and $(x \pm 1$ or $p_1(x \pm 1))$ are squares in \mathbb{N} , then $\{\epsilon_{p_1}, \epsilon_{p_2q}, \sqrt{\epsilon_{p_2q}\epsilon_{p_1p_2q}}\}$ is a F.S.U of \mathbb{K}_1^+ and that of \mathbb{K}_1 is $\{\epsilon_{p_1}, \sqrt{i\epsilon_{p_2q}}, \sqrt{\epsilon_{p_2q}\epsilon_{p_1p_2q}}\}$; moreover, $Q_{\mathbb{K}_1} = 2$.*
2. *If $x \pm 1$ or $p_1(x \pm 1)$ is a square in \mathbb{N} and $a \pm 1$ is not, then $\{\epsilon_{p_1}, \epsilon_{p_2q}, \epsilon_{p_1p_2q}\}$ is a F.S.U of \mathbb{K}_1^+ and that of \mathbb{K}_1 is $\{\epsilon_{p_1}, \epsilon_{p_2q}, \sqrt{i\epsilon_{p_1p_2q}}\}$; moreover, $Q_{\mathbb{K}_1} = 2$.*
3. *If $a \pm 1$ and $2p_1(x \pm 1)$ are squares in \mathbb{N} , then $\{\epsilon_{p_1}, \epsilon_{p_2q}, \sqrt{\epsilon_{p_1p_2q}}\}$ is a F.S.U of \mathbb{K}_1^+ and that of \mathbb{K}_1 is $\{\epsilon_{p_1}, \sqrt{i\epsilon_{p_2q}}, \sqrt{\epsilon_{p_1p_2q}}\}$; moreover, $Q_{\mathbb{K}_1} = 2$.*
4. *If $2p_1(x \pm 1)$ is a square in \mathbb{N} and $a \pm 1$ is not, then $\{\epsilon_{p_1}, \epsilon_{p_2q}, \sqrt{\epsilon_{p_1p_2q}}\}$ is a F.S.U of both of \mathbb{K}_1^+ and \mathbb{K}_1 ; moreover, $Q_{\mathbb{K}_1} = 1$.*
5. *$\{\epsilon_{p_1}, \epsilon_{p_2q}, \epsilon_{p_1p_2q}\}$ is a F.S.U of \mathbb{K}_1^+ , $\{\epsilon_{p_1}, \sqrt{i\epsilon_{p_2q}}, \epsilon_{p_1p_2q}\}$ is a F.S.U of \mathbb{K}_1 and $Q_{\mathbb{K}_1} = 2$ if one of the following assertions is satisfied:*
 - i. $a \pm 1$ and $(p_2(x \pm 1)$ or $2p_2(x \pm 1))$ are squares in \mathbb{N} .
 - ii. $a \pm 1$ and $(q(x \pm 1)$ or $2q(x \pm 1))$ are squares in \mathbb{N} .
6. *$\{\epsilon_{p_1}, \epsilon_{p_2q}, \sqrt{\epsilon_{p_2q}\epsilon_{p_1p_2q}}\}$ is a F.S.U of both of \mathbb{K}_1^+ and \mathbb{K}_1 , and $Q_{\mathbb{K}_1} = 1$ if one of the following assertions is satisfied:*
 - i. $p_2(a \pm 1)$ and $(p_2(x \pm 1)$ or $q(x \pm 1))$ are squares in \mathbb{N} .
 - ii. $2p_2(a \pm 1)$ and $(2p_2(x \pm 1)$ or $2q(x \pm 1))$ are squares in \mathbb{N} .
7. *$\{\epsilon_{p_1}, \epsilon_{p_2q}, \epsilon_{p_1p_2q}\}$ is a F.S.U of \mathbb{K}_1^+ , $\{\epsilon_{p_1}, \epsilon_{p_2q}, \sqrt{i\epsilon_{p_2q}\epsilon_{p_1p_2q}}\}$ is a F.S.U of \mathbb{K}_1 and $Q_{\mathbb{K}_1} = 2$ if one of the following assertions is satisfied:*
 - i. $p_2(a \pm 1)$ and $(2p_2(x \pm 1)$ or $2q(x \pm 1))$ are squares in \mathbb{N} .
 - ii. $2p_2(a \pm 1)$ and $(p_2(x \pm 1)$ or $q(x \pm 1))$ are squares in \mathbb{N} .

Proof. As $N(\epsilon_{p_1}) = -1$, so only ϵ_{p_2q} , $\epsilon_{p_1p_2q}$ and $\epsilon_{p_2q}\epsilon_{p_1p_2q}$ can be squares in \mathbb{K}_1^+ .

1. Let $\epsilon_{p_2q} = a + b\sqrt{p_2q}$, where a and b are integers of different parities satisfying $a^2 - 1 = p_2qb^2$, hence $(a \pm 1)(a \mp 1) = p_2qb^2$ and \gcd of $a \pm 1$, $a \mp 1$ divides 2. Thus by the decomposition uniqueness in \mathbb{Z} and by Lemma 2.2, there exist b_1, b_2 in \mathbb{Z} such that:

- a. If $a \pm 1$ is a square in \mathbb{N} , then $\begin{cases} a \pm 1 = b_1^2, \\ a \mp 1 = b_2^2 p_2q, \end{cases}$ so $\sqrt{\epsilon_{p_2q}} = \frac{1}{\sqrt{2}}(b_1 + b_2\sqrt{p_2q})$, and thus ϵ_{p_2q} is not a square in \mathbb{K}_1 but $2\epsilon_{p_2q}$ is.
- b. If $p_2(a \pm 1)$ is a square in \mathbb{N} , then $\begin{cases} a \pm 1 = b_1^2 p_2 \\ a \mp 1 = b_2^2 q, \end{cases}$ so $\sqrt{\epsilon_{p_2q}} = \frac{1}{\sqrt{2}}(b_1\sqrt{p_2} + b_2\sqrt{q})$, thus ϵ_{p_2q} and $2\epsilon_{p_2q}$ are not squares in \mathbb{K}_1 , but $2p_2\epsilon_{p_2q}$ and $2q\epsilon_{p_2q}$ are.
- c. If $2p_2(a \pm 1)$ is a square in \mathbb{N} , then $\begin{cases} a \pm 1 = 2b_1^2 p_2 \\ a \mp 1 = 2b_2^2 q, \end{cases}$ so $\sqrt{\epsilon_{p_2q}} = b_1\sqrt{p_2} + b_2\sqrt{q}$, thus ϵ_{p_2q} is not a square in \mathbb{K}_1 , but $p_2\epsilon_{p_2q}$ and $q\epsilon_{p_2q}$ are.

2. Similarly, let $\epsilon_{p_1p_2q} = x + y\sqrt{p_1p_2q}$, where x and y are integers of different parities satisfying $x^2 - 1 = p_1p_2qy^2$, hence $(x \pm 1)(x \mp 1) = p_1p_2qy^2$ and the \gcd of $x \pm 1$, $x \mp 1$ divides 2. By Lemma 2.2, $2(x \pm 1)$ and $2p_1p_2q(x \pm 1)$ are not squares in \mathbb{N} . Thus, the decomposition uniqueness in \mathbb{Z} enables us to distinguish the following cases:

- i. If $x \pm 1$ is a square in \mathbb{N} , then $\begin{cases} x \pm 1 = y_1^2 \\ x \mp 1 = y_2^2 p_1p_2q, \end{cases}$ so $\sqrt{2\epsilon_{p_1p_2q}} = y_1 + y_2\sqrt{p_1p_2q}$, hence $\epsilon_{p_1p_2q}$ is not a square in \mathbb{K}_1^+ , but $2\epsilon_{p_1p_2q}$ is.
- ii. If $p_1(x \pm 1)$ is a square in \mathbb{N} , then $\begin{cases} x \pm 1 = y_1^2 p_1 \\ x \mp 1 = y_2^2 p_2q, \end{cases}$ so $\sqrt{2\epsilon_{p_1p_2q}} = y_1\sqrt{p_1} + y_2\sqrt{p_2q}$, hence $\epsilon_{p_1p_2q}$ is not a square in \mathbb{K}_1^+ , but $2\epsilon_{p_1p_2q}$ is.
- iii. If $2p_1(x \pm 1)$ is a square in \mathbb{N} , then $\begin{cases} x \pm 1 = 2y_1^2 p_1 \\ x \mp 1 = 2y_2^2 p_2q, \end{cases}$ so $\sqrt{\epsilon_{p_1p_2q}} = y_1\sqrt{p_1} + y_2\sqrt{p_2q}$, hence $\epsilon_{p_1p_2q}$ is a square in \mathbb{K}_1^+ .
- iv. If $p_2(x \pm 1)$ is a square in \mathbb{N} , then $\sqrt{2\epsilon_{p_1p_2q}} = y_1\sqrt{p_2} + y_2\sqrt{p_1q}$, hence $\epsilon_{p_1p_2q}$, $2\epsilon_{p_1p_2q}$ are not squares in \mathbb{K}_1^+ , but $2p_2\epsilon_{p_1p_2q}$ is.
- v. If $2p_2(x \pm 1)$ is a square in \mathbb{N} , then $\sqrt{\epsilon_{p_1p_2q}} = y_1\sqrt{p_2} + y_2\sqrt{p_1q}$, hence $\epsilon_{p_1p_2q}$ is not a square in \mathbb{K}_1^+ , but $p_2\epsilon_{p_1p_2q}$ is.
- vi. If $q(x \pm 1)$ is a square in \mathbb{N} , then $\sqrt{2\epsilon_{p_1p_2q}} = y_1\sqrt{q} + y_2\sqrt{p_1p_2}$, hence $\epsilon_{p_1p_2q}$, $2\epsilon_{p_1p_2q}$ are not squares in \mathbb{K}_1^+ , but $2q\epsilon_{p_1p_2q}$ is.
- vii. If $2q(x \pm 1)$ is a square in \mathbb{N} , then $\sqrt{\epsilon_{p_1p_2q}} = y_1\sqrt{q} + y_2\sqrt{p_1p_2}$, hence $\epsilon_{p_1p_2q}$ is not a square in \mathbb{K}_1^+ , but $q\epsilon_{p_1p_2q}$ is.

Consequently, if $a \pm 1$ and $(x \pm 1$ or $p_1(x \pm 1))$ are squares in \mathbb{N} , then $2\epsilon_{p_2q}$, $2\epsilon_{p_1p_2q}$ are squares in \mathbb{K}_1^+ ; hence $\epsilon_{p_2q}\epsilon_{p_1p_2q}$ is a square in \mathbb{K}_1^+ . Thus $\{\epsilon_{p_1}, \epsilon_{p_2q}, \sqrt{\epsilon_{p_2q}\epsilon_{p_1p_2q}}\}$ is a F.S.U of \mathbb{K}_1^+ , and as $2\epsilon_{p_2q}$ is a square in \mathbb{K}_1^+ , so by Lemma 2.1 $\{\epsilon_{p_1}, \sqrt{i\epsilon_{p_2q}}, \sqrt{\epsilon_{p_2q}\epsilon_{p_1p_2q}}\}$ is a F.S.U de \mathbb{K}_1 . Thus $Q_{\mathbb{K}_1} = 2$.

The other cases are similarly treated. \square

3.2. F.S.U of the field \mathbb{K}_2 . As p_1 and p_2 play symmetric roles, so we similarly get the F.S.U of $\mathbb{K}_2 = \mathbb{k}(\sqrt{p_2}) = \mathbb{Q}(\sqrt{p_2}, \sqrt{p_1q}, i)$.

Proposition 3.2. *Keep the previous notations and put $\epsilon_{p_1q} = a + b\sqrt{p_1q}$.*

1. *If $a \pm 1$ and $(x \pm 1$ or $p_2(x \pm 1))$ are squares in \mathbb{N} , then $\{\epsilon_{p_2}, \epsilon_{p_1q}, \sqrt{\epsilon_{p_1q}\epsilon_{p_1p_2q}}\}$ is a F.S.U of \mathbb{K}_2^+ and that of \mathbb{K}_2 is $\{\epsilon_{p_2}, \sqrt{i\epsilon_{p_1q}}, \sqrt{\epsilon_{p_1q}\epsilon_{p_1p_2q}}\}$; moreover, $Q_{\mathbb{K}_2} = 2$.*
2. *If $x \pm 1$ or $p_2(x \pm 1)$ is a square in \mathbb{N} and $a \pm 1$ is not, then $\{\epsilon_{p_2}, \epsilon_{p_1q}, \epsilon_{p_1p_2q}\}$ is a F.S.U of \mathbb{K}_2^+ and that of \mathbb{K}_2 is $\{\epsilon_{p_2}, \epsilon_{p_1q}, \sqrt{i\epsilon_{p_1p_2q}}\}$; moreover, $Q_{\mathbb{K}_2} = 2$.*
3. *If $a \pm 1$ and $2p_2(x \pm 1)$ are squares in \mathbb{N} , then $\{\epsilon_{p_2}, \epsilon_{p_1q}, \sqrt{\epsilon_{p_1p_2q}}\}$ is a F.S.U of \mathbb{K}_2^+ and that of \mathbb{K}_2 is $\{\epsilon_{p_2}, \sqrt{i\epsilon_{p_1q}}, \sqrt{\epsilon_{p_1p_2q}}\}$; moreover, $Q_{\mathbb{K}_2} = 2$.*
4. *If $2p_2(x \pm 1)$ is a square in \mathbb{N} and $a \pm 1$ is not, then $\{\epsilon_{p_2}, \epsilon_{p_1q}, \sqrt{\epsilon_{p_1p_2q}}\}$ is a F.S.U of both of \mathbb{K}_2^+ and \mathbb{K}_2 ; moreover, $Q_{\mathbb{K}_2} = 1$.*
5. *$\{\epsilon_{p_2}, \epsilon_{p_1q}, \epsilon_{p_1p_2q}\}$ is a F.S.U of \mathbb{K}_2^+ , $\{\epsilon_{p_2}, \sqrt{i\epsilon_{p_1q}}, \epsilon_{p_1p_2q}\}$ is a F.S.U of \mathbb{K}_2 and $Q_{\mathbb{K}_2} = 2$ if one of the following assertions is satisfied:*
 - i. $a \pm 1$ and $(p_1(x \pm 1)$ or $2p_1(x \pm 1))$ are squares in \mathbb{N} .
 - ii. $a \pm 1$ and $(q(x \pm 1)$ or $2q(x \pm 1))$ are squares in \mathbb{N} .
6. *$\{\epsilon_{p_2}, \epsilon_{p_1q}, \sqrt{\epsilon_{p_1q}\epsilon_{p_1p_2q}}\}$ is a F.S.U of both of \mathbb{K}_2^+ and \mathbb{K}_2 , and $Q_{\mathbb{K}_2} = 1$, if one of the following assertions is satisfied:*
 - i. $p_1(a \pm 1)$ and $(p_1(x \pm 1)$ or $q(x \pm 1))$ are squares in \mathbb{N} .
 - ii. $2p_1(a \pm 1)$ and $(2p_1(x \pm 1)$ or $2q(x \pm 1))$ are squares in \mathbb{N} .
7. *$\{\epsilon_{p_2}, \epsilon_{p_1q}, \epsilon_{p_1p_2q}\}$ is a F.S.U of \mathbb{K}_2^+ , $\{\epsilon_{p_2}, \epsilon_{p_1q}, \sqrt{i\epsilon_{p_1q}\epsilon_{p_1p_2q}}\}$ is a F.S.U of \mathbb{K}_2 and $Q_{\mathbb{K}_2} = 2$ if one of the following assertions is satisfied:*
 - i. $p_1(a \pm 1)$ and $(2p_1(x \pm 1)$ or $2q(x \pm 1))$ are squares in \mathbb{N} .
 - ii. $2p_1(a \pm 1)$ and $(p_1(x \pm 1)$ or $q(x \pm 1))$ are squares in \mathbb{N} .

3.3. F.S.U of the field \mathbb{K}_3 . Let $\mathbb{K}_3 = \mathbb{k}(\sqrt{q}) = \mathbb{Q}(\sqrt{q}, \sqrt{p_1p_2}, i)$.

Proposition 3.3. *Keep the previous notations and assume $N(\epsilon_{p_1p_2}) = 1$. Then $Q_{\mathbb{K}_3} = 2$ and we have:*

1. *If $2q(x \pm 1)$ is a square in \mathbb{N} , then $\{\epsilon_q, \epsilon_{p_1p_2}, \sqrt{\epsilon_{p_1p_2q}}\}$ is a F.S.U of \mathbb{K}_3^+ and that of \mathbb{K}_3 is $\{\epsilon_{p_1p_2}, \sqrt{\epsilon_{p_1p_2q}}, \sqrt{i\epsilon_q}\}$.*
2. *If $x \pm 1$ or $q(x \pm 1)$ is a square in \mathbb{N} , then $\{\epsilon_q, \epsilon_{p_1p_2}, \sqrt{\epsilon_q\epsilon_{p_1p_2q}}\}$ is a F.S.U of \mathbb{K}_3^+ and that of \mathbb{K}_3 is a $\{\epsilon_{p_1p_2}, \sqrt{\epsilon_q\epsilon_{p_1p_2q}}, \sqrt{i\epsilon_q}\}$.*
3. *If $p_1(x \pm 1)$ or $p_2(x \pm 1)$ is a square in \mathbb{N} , then $\{\epsilon_q, \epsilon_{p_1p_2}, \sqrt{\epsilon_q\epsilon_{p_1p_2}\epsilon_{p_1p_2q}}\}$ is a F.S.U of \mathbb{K}_3^+ and that of \mathbb{K}_3 is $\{\epsilon_{p_1p_2}, \sqrt{\epsilon_q\epsilon_{p_1p_2}\epsilon_{p_1p_2q}}, \sqrt{i\epsilon_q}\}$.*
4. *If $2p_1(x \pm 1)$ or $2p_2(x \pm 1)$ is a square in \mathbb{N} , then $\{\epsilon_q, \epsilon_{p_1p_2}, \sqrt{\epsilon_{p_1p_2}\epsilon_{p_1p_2q}}\}$ is a F.S.U of \mathbb{K}_3^+ and that of \mathbb{K}_3 is $\{\epsilon_{p_1p_2}, \sqrt{\epsilon_{p_1p_2}\epsilon_{p_1p_2q}}, \sqrt{i\epsilon_q}\}$.*

Proof. As the norms of ϵ_q , $\epsilon_{p_1p_2}$ and $\epsilon_{p_1p_2q}$ are equal to 1, then a F.S.U of \mathbb{K}_3^+ is a system consisting of three elements chosen from B' , where $B' = B \cup \{\sqrt{\mu}/\mu \in B \text{ et } \sqrt{\mu} \in \mathbb{K}^+\}$, with

$$B = \{\epsilon_q, \epsilon_{p_1p_2}, \epsilon_{p_1p_2q}, \epsilon_q\epsilon_{p_1p_2}, \epsilon_q\epsilon_{p_1p_2q}, \epsilon_{p_1p_2}\epsilon_{p_1p_2q}, \epsilon_q\epsilon_{p_1p_2}\epsilon_{p_1p_2q}\}.$$

According to Lemma 2.3, ϵ_q is not a square in $\mathbb{Q}(\sqrt{q})$, but $2\epsilon_q$ is.

Put $\epsilon_{p_1 p_2} = a + b\sqrt{p_1 p_2}$, then $a^2 - 1 = b^2 p_1 p_2$. Hence by Lemmas 2.2 and 2.6 we get that only $2p_1(a \pm 1)$ is a square in \mathbb{N} , so $\epsilon_{p_1 p_2}$ is not a square in \mathbb{K}_3^+ , but $p_1 \epsilon_{p_1 p_2}$, $p_2 \epsilon_{p_1 p_2}$ are.

Let $\epsilon_{p_1 p_2 q} = x + y\sqrt{p_1 p_2 q}$, then proceeding as in the proof of Proposition 3.1, we get:

- If $x \pm 1$ is a square in \mathbb{N} , then $\sqrt{2\epsilon_{p_1 p_2 q}} = y_1 + y_2\sqrt{p_1 p_2 q}$, so $\epsilon_{p_1 p_2 q}$ is not a square in \mathbb{K}_3^+ , but $2\epsilon_{p_1 p_2 q}$ is.
- If $p_1(x \pm 1)$ is a square in \mathbb{N} , then $\sqrt{2\epsilon_{p_1 p_2 q}} = y_1\sqrt{p_1} + y_2\sqrt{p_2 q}$, so $\epsilon_{p_1 p_2 q}$ and $2\epsilon_{p_1 p_2 q}$ are not squares in \mathbb{K}_3^+ , but $2p_1\epsilon_{p_1 p_2 q}$ and $2p_2 q\epsilon_{p_1 p_2 q}$ are.
- If $2p_1(x \pm 1)$ is a square in \mathbb{N} , then $\sqrt{\epsilon_{p_1 p_2 q}} = y_1\sqrt{p_1} + y_2\sqrt{p_2 q}$, so $\epsilon_{p_1 p_2 q}$ is not a square in \mathbb{K}_3^+ , but $p_1\epsilon_{p_1 p_2 q}$ and $p_2 q\epsilon_{p_1 p_2 q}$ are.
- If $p_2(x \pm 1)$ is a square in \mathbb{N} , then $\sqrt{2\epsilon_{p_1 p_2 q}} = y_1\sqrt{p_2} + y_2\sqrt{p_1 q}$, so $\epsilon_{p_1 p_2 q}$ and $2\epsilon_{p_1 p_2 q}$ are not squares in \mathbb{K}_3^+ , but $2p_2\epsilon_{p_1 p_2 q}$ and $2p_1 q\epsilon_{p_1 p_2 q}$ are.
- If $2p_2(x \pm 1)$ is a square in \mathbb{N} , then $\sqrt{\epsilon_{p_1 p_2 q}} = y_1\sqrt{p_2} + y_2\sqrt{p_1 q}$, so $\epsilon_{p_1 p_2 q}$ is not a square in \mathbb{K}_3^+ , but $p_2\epsilon_{p_1 p_2 q}$, $p_1 q\epsilon_{p_1 p_2 q}$ are.
- If $q(x \pm 1)$ is a square in \mathbb{N} , then $\sqrt{2\epsilon_{p_1 p_2 q}} = y_1\sqrt{q} + y_2\sqrt{p_1 p_2}$, so $\epsilon_{p_1 p_2 q}$ is not a square in \mathbb{K}_3^+ , but $2\epsilon_{p_1 p_2 q}$ is.
- If $2q(x \pm 1)$ is a square in \mathbb{N} , then $\sqrt{\epsilon_{p_1 p_2 q}} = y_1\sqrt{q} + y_2\sqrt{p_1 p_2}$, so $\epsilon_{p_1 p_2 q}$ is a square in \mathbb{K}_3^+ .

Consequently, we have:

- If $x \pm 1$ or $q(x \pm 1)$ is a square in \mathbb{N} , then $\epsilon_q \epsilon_{p_1 p_2 q}$ is a square in \mathbb{K}_3^+ , so $\{\epsilon_q, \epsilon_{p_1 p_2}, \sqrt{\epsilon_q \epsilon_{p_1 p_2 q}}\}$ is a F.S.U of \mathbb{K}_3^+ . As $2\epsilon_q$ is a square in \mathbb{K}_3^+ , then by Lemma 2.1, $\{\epsilon_{p_1 p_2}, \sqrt{\epsilon_q \epsilon_{p_1 p_2 q}}, \sqrt{i\epsilon_q}\}$ is a F.S.U of \mathbb{K}_3 .
- If $2q(x \pm 1)$ is a square in \mathbb{N} , then $\epsilon_{p_1 p_2 q}$ is a square in \mathbb{K}_3^+ , hence $\{\epsilon_q, \epsilon_{p_1 p_2}, \sqrt{\epsilon_{p_1 p_2 q}}\}$ is a F.S.U of \mathbb{K}_3^+ , and by Lemma 2.1 $\{\epsilon_{p_1 p_2}, \sqrt{\epsilon_{p_1 p_2 q}}, \sqrt{i\epsilon_q}\}$ is a F.S.U of \mathbb{K}_3 .
- If $p_1(x \pm 1)$ is a square in \mathbb{N} , then $2p_1\epsilon_{p_1 p_2 q}$ is a square in \mathbb{K}_3^+ , and since $p_1\epsilon_{p_1 p_2}$ and $2\epsilon_q$ are too, so $\epsilon_q \epsilon_{p_1 p_2} \epsilon_{p_1 p_2 q}$ is a square in \mathbb{K}_3^+ , hence $\{\epsilon_q, \epsilon_{p_1 p_2}, \sqrt{\epsilon_q \epsilon_{p_1 p_2} \epsilon_{p_1 p_2 q}}\}$ is a F.S.U of \mathbb{K}_3^+ . As $2\epsilon_q$ is a square in \mathbb{K}_3^+ , then by Lemma 2.1 $\{\epsilon_{p_1 p_2}, \sqrt{\epsilon_q \epsilon_{p_1 p_2} \epsilon_{p_1 p_2 q}}, \sqrt{i\epsilon_q}\}$ is a F.S.U of \mathbb{K}_3 .
- If $2p_2(x \pm 1)$ is a square in \mathbb{N} , then $p_2\epsilon_{p_1 p_2 q}$ is a square in \mathbb{K}_3^+ , and since $p_2\epsilon_{p_1 p_2}$ is too, so $\epsilon_{p_1 p_2} \epsilon_{p_1 p_2 q}$ is a square in \mathbb{K}_3^+ . This yields that $\{\epsilon_q, \epsilon_{p_1 p_2}, \sqrt{\epsilon_{p_1 p_2} \epsilon_{p_1 p_2 q}}\}$ is a F.S.U of \mathbb{K}_3^+ . On the other hand, $2\epsilon_q$ is a square in \mathbb{K}_3^+ , then by Lemma 2.1, $\{\epsilon_{p_1 p_2}, \sqrt{\epsilon_{p_1 p_2} \epsilon_{p_1 p_2 q}}, \sqrt{i\epsilon_q}\}$ is a F.S.U of \mathbb{K}_3 .

The other cases are similarly proved. \square

Proposition 3.4. *Keep the previous notations and assume $N(\epsilon_{p_1 p_2}) = -1$. Then $Q_{\mathbb{K}_3} = 2$ and we have:*

- If $2q(x \pm 1)$ is a square in \mathbb{N} , then $\{\epsilon_q, \epsilon_{p_1 p_2}, \sqrt{\epsilon_{p_1 p_2 q}}\}$ is a F.S.U of \mathbb{K}_3^+ and that of \mathbb{K}_3 is $\{\epsilon_{p_1 p_2}, \sqrt{\epsilon_{p_1 p_2 q}}, \sqrt{i\epsilon_q}\}$.

2. If $x \pm 1$ or $q(x \pm 1)$ is a square in \mathbb{N} , then $\{\epsilon_q, \epsilon_{p_1 p_2}, \sqrt{\epsilon_q \epsilon_{p_1 p_2 q}}\}$ is a F.S.U of \mathbb{K}_3^+ and that of \mathbb{K}_3 is $\{\epsilon_{p_1 p_2}, \sqrt{\epsilon_q \epsilon_{p_1 p_2 q}}, \sqrt{i \epsilon_q}\}$.
3. In the other cases $\{\epsilon_q, \epsilon_{p_1 p_2}, \epsilon_{p_1 p_2 q}\}$ is a F.S.U of \mathbb{K}_3^+ and that of \mathbb{K}_3 is $\{\epsilon_{p_1 p_2}, \epsilon_{p_1 p_2 q}, \sqrt{i \epsilon_q}\}$.

Proof. As $N(\epsilon_{p_1 p_2} 1) = -$, so by Lemma 2.3, only $\epsilon_{p_1 p_2 q}$ and $\epsilon_q \epsilon_{p_1 p_2 q}$ can be squares in \mathbb{K}_3^+ . Proceeding as above, we get the results. \square

4. The ambiguous classes of $\mathbb{k}/\mathbb{Q}(i)$

Let $F = \mathbb{Q}(i)$ and $\mathbb{k} = \mathbb{Q}(\sqrt{p_1 p_2 q}, i)$. We denote by $Am(\mathbb{k}/F)$ the group of the ambiguous classes of \mathbb{k}/F and by $Am_s(\mathbb{k}/F)$ the subgroup of $Am(\mathbb{k}/F)$ generated by the strongly ambiguous classes. As $p_1 \equiv p_2 \equiv 1 \pmod{4}$, so there exist e, f, g and h in \mathbb{N} such that $p_1 = e^2 + 4f^2 = \pi_1 \pi_2$ and $p_2 = g^2 + 4h^2 = \pi_3 \pi_4$. Put $\pi_1 = e + 2if$, $\pi_2 = e - 2if$, $\pi_3 = g + 2ih$ and $\pi_4 = g - 2ih$. Let \mathcal{H}_j (resp. \mathcal{Q}) be the prime ideal of \mathbb{k} above π_j (resp. q), where $j \in \{1, 2, 3, 4\}$. It is easy to see that $\mathcal{H}_j^2 = (\pi_j)$ and $\mathcal{Q}^2 = (q)$. Therefore $[\mathcal{Q}]$ and $[\mathcal{H}_j]$ are in $Am_s(\mathbb{k}/F)$, for all $j \in \{1, 2, 3, 4\}$. Keep the notation $\epsilon_{p_1 p_2 q} = x + y\sqrt{p_1 p_2 q}$. In this section, we will determine generators of $Am_s(\mathbb{k}/F)$ and $Am(\mathbb{k}/F)$. Let us first prove the following result.

Lemma 4.1. *Consider the prime ideals \mathcal{H}_j of \mathbb{k} , $1 \leq j \leq 4$.*

1. If $x \pm 1$ is a square in \mathbb{N} , then $|\langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_3], [\mathcal{H}_4] \rangle| = 16$.
2. If $q(x \pm 1)$ or $2q(x \pm 1)$ is a square in \mathbb{N} , then $|\langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_3] \rangle| = |\langle [\mathcal{H}_1], [\mathcal{H}_3], [\mathcal{H}_4] \rangle| = 8$.
3. If $p_1(x \pm 1)$ or $2p_1(x \pm 1)$ is a square in \mathbb{N} , then $|\langle [\mathcal{H}_1], [\mathcal{H}_3], [\mathcal{H}_4] \rangle| = 8$.
4. If $p_2(x \pm 1)$ or $2p_2(x \pm 1)$ is a square in \mathbb{N} , then $|\langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_3] \rangle| = 8$.

Proof. Since $\mathcal{H}_j^2 = (\pi_j)$, for all $1 \leq j \leq 2$, and since also $\sqrt{e^2 + (2f)^2} = \sqrt{p_1} \notin \mathbb{Q}(\sqrt{p_1 p_2 q})$ and $\sqrt{g^2 + (2h)^2} = \sqrt{p_2} \notin \mathbb{Q}(\sqrt{p_1 p_2 q})$, so, according to [4, Proposition 1], \mathcal{H}_j are not principal in \mathbb{k} .

1. If $x \pm 1$ is a square in \mathbb{N} , then $\epsilon_{p_1 p_1 q}$ is not a F.S.U of \mathbb{k} (by Lemma 2.5) and for all prime ℓ dividing $p_1 p_2 q$, $\ell(x + 1)$, $\ell(x - 1)$, $2\ell(x + 1)$, $2\ell(x - 1)$ are not squares in \mathbb{N} . We have:

$(\mathcal{H}_1 \mathcal{H}_2)^2 = (p_1)$, $(\mathcal{H}_3 \mathcal{H}_4)^2 = (p_2)$ and $\mathcal{Q}^2 = (q)$, hence according to [4, Proposition 2], $\mathcal{H}_1 \mathcal{H}_2$, $\mathcal{H}_3 \mathcal{H}_4$ and \mathcal{Q} are not principal in \mathbb{k} .

For $i \in \{1, 2\}$ and $j \in \{3, 4\}$, $\mathcal{H}_i \mathcal{H}_j$ is not principal in \mathbb{k} , in fact, $(\mathcal{H}_i \mathcal{H}_j)^2 = (\pi_i \pi_j)$ and as $\pi_1 \pi_3 = (eg - 4fh) + 2i(eh + fg)$, $\pi_1 \pi_4 = (eg + 4fh) - 2i(eh - fg)$, $\pi_2 \pi_3 = (eg + 4fh) + 2i(eh - fg)$, $\pi_2 \pi_4 = (eg - 4fh) - 2i(eh + fg)$, and also $\sqrt{(eg - 4fh)^2 + 4(eh + fg)^2} = \sqrt{(eg + 4fh)^2 + 4(eh - fg)^2} = \sqrt{p_1 p_2} \notin \mathbb{Q}(\sqrt{d})$, hence [4, Proposition 1] yields the result.

For $i \in \{1, 2\}$ (resp. $j \in \{3, 4\}$), the ideal $\mathcal{H}_i \mathcal{H}_3 \mathcal{H}_4$ (resp. $\mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_j$) is not principal, since $(\mathcal{H}_i \mathcal{H}_3 \mathcal{H}_4)^2 = (\pi_i p_2)$ (resp. $(\mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_j)^2 = (p_1 \pi_j)$), and as $\sqrt{((p_2 e)^2 + 4(p_2 f)^2)} = p_2 \sqrt{p_1} \notin \mathbb{Q}(\sqrt{d})$ (resp. $\sqrt{((p_1 g)^2 + 4(p_1 h)^2)} = p_1 \sqrt{p_2} \notin \mathbb{Q}(\sqrt{d})$), then [4, Proposition 1] states the result.

Finally, as $(\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\mathcal{H}_4)^2 = (p_1p_2)$, so [4, Remark 1] implies that $\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\mathcal{H}_4$ is not principal in \mathbb{k} . Note at the end that $[\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\mathcal{H}_4] = [\mathcal{Q}]$, since $[\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\mathcal{H}_4\mathcal{Q}] = [(\sqrt{p_1p_2q})] = 1$.

2. If $q(x \pm 1)$ or $2q(x \pm 1)$ are not squares in \mathbb{N} , then according to [4, Proposition 2] $[\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\mathcal{H}_4] = [\mathcal{Q}]$ is principal in \mathbb{k} . Hence the result.

The other assertions are similarly proved. \square

Determine now generators of $\text{Am}_s(\mathbb{k}/F)$ and $\text{Am}(\mathbb{k}/F)$. According to the ambiguous class number formula (see [10]), the genus number, $[(\mathbb{k}/F)^* : \mathbb{k}]$, is given by:

$$|\text{Am}(\mathbb{k}/F)| = [(\mathbb{k}/F)^* : \mathbb{k}] = \frac{h(F)2^{t-1}}{[E_F : E_F \cap N_{\mathbb{k}/F}(\mathbb{k}^\times)]}, \quad (1)$$

where $h(F)$ is the class number of F and t is the number of finite and infinite primes of F ramified in \mathbb{k}/F . Moreover as the class number of F is equal to 1, so the formula (1) yields that

$$|\text{Am}(\mathbb{k}/F)| = [(\mathbb{k}/F)^* : \mathbb{k}] = 2^r, \quad (2)$$

where $r = \text{rank Cl}_2(\mathbb{k}) = t - e - 1$ and $2^e = [E_F : E_F \cap N_{\mathbb{k}/F}(\mathbb{k}^\times)]$ (see for example [19]). The relation between $|\text{Am}(\mathbb{k}/F)|$ and $|\text{Am}_s(\mathbb{k}/F)|$ is given by the following formula (see for example [11]):

$$\frac{|\text{Am}(\mathbb{k}/F)|}{|\text{Am}_s(\mathbb{k}/F)|} = [E_F \cap N_{\mathbb{k}/F}(\mathbb{k}^\times) : N_{\mathbb{k}/F}(E_{\mathbb{k}})]. \quad (3)$$

To continue, we need the following lemma.

Lemma 4.2 ([19]). *Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod{4}$ be different primes, $F = \mathbb{Q}(i)$ and $\mathbb{k} = \mathbb{Q}(\sqrt{p_1p_2q}, i)$.*

1. *If $p_1 \equiv p_2 \equiv 1 \pmod{8}$, then i is a norm in \mathbb{k}/F .*
2. *If $p_1 \equiv 5$ or $p_2 \equiv 5 \pmod{8}$, then i is not a norm in \mathbb{k}/F .*

Proposition 4.3. *Let $(\mathbb{k}/F)^*$ denote the relative genus field of \mathbb{k}/F .*

1. $\mathbb{k}^{(*)} \subsetneq (\mathbb{k}/F)^*$ and $[(\mathbb{k}/F)^* : \mathbb{k}^{(*)}] = 2$ or 4.
2. Assume $p_1 \equiv p_2 \equiv 1 \pmod{8}$.
 - i. *If $x \pm 1$ is a square in \mathbb{N} , then*
 $\text{Am}(\mathbb{k}/\mathbb{Q}(i)) = \text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_3], [\mathcal{H}_4] \rangle$.
 - ii. *Else, there exist an unambiguous ideal \mathcal{I} in $\mathbb{k}/\mathbb{Q}(i)$ of order 2 such that*
 $\text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_3] \rangle$ or $\langle [\mathcal{H}_1], [\mathcal{H}_3], [\mathcal{H}_4] \rangle$ and
 $\text{Am}(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_3], [\mathcal{I}] \rangle$ or $\langle [\mathcal{H}_1], [\mathcal{H}_3], [\mathcal{H}_4], [\mathcal{I}] \rangle$.
3. Assume $p_1 \equiv 5$ or $p_2 \equiv 5 \pmod{8}$, then neither $x+1$ nor $x-1$ is a square in \mathbb{N} and $\text{Am}(\mathbb{k}/\mathbb{Q}(i)) = \text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_3] \rangle$ or $\langle [\mathcal{H}_1], [\mathcal{H}_3], [\mathcal{H}_4] \rangle$.

Proof. 1. As $\mathbb{k} = \mathbb{Q}(\sqrt{p_1p_2q}, i)$, so $[\mathbb{k}^{(*)} : \mathbb{k}] = 4$. Moreover, according to [19, Proposition 2, p. 90], $r = \text{rank Cl}_2(\mathbb{k}) = 4$ if $p_1 \equiv p_2 \equiv 1 \pmod{8}$ and $r = \text{rank Cl}_2(\mathbb{k}) = 3$ if $p_1 \equiv 5$ or $p_2 \equiv 5 \pmod{8}$, so $[(\mathbb{k}/F)^* : \mathbb{k}] = 8$ or 16. Hence $[(\mathbb{k}/F)^* : \mathbb{k}^{(*)}] = 2$ or 4, and the result derived.

2. Assume $p_1 \equiv p_2 \equiv 1 \pmod{8}$, hence i is a norm in $\mathbb{k}/\mathbb{Q}(i)$ (Lemma 4.2), thus Formula (3) yields that

$$\begin{aligned} \frac{|\text{Am}(\mathbb{k}/\mathbb{Q}(i))|}{|\text{Am}_s(\mathbb{k}/\mathbb{Q}(i))|} &= [E_{\mathbb{Q}(i)} \cap N_{\mathbb{k}/\mathbb{Q}(i)}(\mathbb{k}^\times) : N_{\mathbb{k}/\mathbb{Q}(i)}(E_{\mathbb{k}})] \\ &= \begin{cases} 1 & \text{if } x \pm 1 \text{ is a square in } \mathbb{N}, \\ 2 & \text{if not,} \end{cases} \end{aligned}$$

since in the case where $x \pm 1$ is a square in \mathbb{N} , we have $E_{\mathbb{k}} = \langle i, \sqrt{i\epsilon_{p_1 p_2 q}} \rangle$, hence $[E_{\mathbb{Q}(i)} \cap N_{\mathbb{k}/\mathbb{Q}(i)}(\mathbb{k}^\times) : N_{\mathbb{k}/\mathbb{Q}(i)}(E_{\mathbb{k}})] = [\langle i \rangle : \langle i \rangle] = 1$, and if not we have $E_{\mathbb{k}} = \langle i, \epsilon_{p_1 p_2 q} \rangle$, hence $[E_{\mathbb{Q}(i)} \cap N_{\mathbb{k}/\mathbb{Q}(i)}(\mathbb{k}^\times) : N_{\mathbb{k}/\mathbb{Q}(i)}(E_{\mathbb{k}})] = [\langle i \rangle : \langle -1 \rangle] = 2$.

On the other hand, as $p_1 \equiv p_2 \equiv 1 \pmod{8}$, so according to [19, Proposition 2, p. 90], $r = 4$. Therefore $|\text{Am}(\mathbb{k}/\mathbb{Q}(i))| = 2^4$.

i. If $x \pm 1$ is a square in \mathbb{N} , then $\text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \text{Am}(\mathbb{k}/\mathbb{Q}(i))$, hence by Lemma 4.1 we get $\text{Am}(\mathbb{k}/\mathbb{Q}(i)) = \text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_3], [\mathcal{H}_4] \rangle$.

ii. If $x + 1$ and $x - 1$ are not squares in \mathbb{N} , then

$$|\text{Am}(\mathbb{k}/\mathbb{Q}(i))| = 2|\text{Am}_s(\mathbb{k}/\mathbb{Q}(i))| = 16,$$

hence Lemma 4.1 yields that $\text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_3] \rangle$ or $\langle [\mathcal{H}_1], [\mathcal{H}_3], [\mathcal{H}_4] \rangle$.

Consequently, there exist an unambiguous ideal \mathcal{I} in \mathbb{k}/F of order 2 such that

$$\text{Am}(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_3], [\mathcal{I}] \rangle \text{ or } \langle [\mathcal{H}_1], [\mathcal{H}_3], [\mathcal{H}_4], [\mathcal{I}] \rangle.$$

By Chebotarev theorem, \mathcal{I} can always be chosen as a prime ideal of \mathbb{k} above a prime l in \mathbb{Q} , which splits completely in \mathbb{k} .

3. Assume $p_1 \equiv 5$ or $p_2 \equiv 5 \pmod{8}$, hence i is not a norm in $\mathbb{k}/\mathbb{Q}(i)$ (Lemma 4.2) and $x + 1$, $x - 1$ are not squares in \mathbb{N} , for if $x \pm 1$ is a square in \mathbb{N} , then the Legendre symbol implies that

$$1 = \left(\frac{x \pm 1}{p_j} \right) = \left(\frac{x \mp 1 \pm 2}{p_j} \right) = \left(\frac{2}{p_j} \right) \quad \text{for all } j \in \{1, 2\},$$

which is absurd. Thus $|\text{Am}(\mathbb{k}/\mathbb{Q}(i))| = 2^3$ and

$$\frac{|\text{Am}(\mathbb{k}/\mathbb{Q}(i))|}{|\text{Am}_s(\mathbb{k}/\mathbb{Q}(i))|} = [E_{\mathbb{Q}(i)} \cap N_{\mathbb{k}/\mathbb{Q}(i)}(\mathbb{k}^\times) : N_{\mathbb{k}/\mathbb{Q}(i)}(E_{\mathbb{k}})] = 1.$$

Hence by Lemma 4.1 we get $\text{Am}(\mathbb{k}/\mathbb{Q}(i)) = \text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_3] \rangle$ or $\langle [\mathcal{H}_1], [\mathcal{H}_3], [\mathcal{H}_4] \rangle$. This completes the proof. \square

5. Capitulation

In this section, we will determine the classes of $\mathbf{Cl}_2(\mathbb{k})$, the 2-class group of \mathbb{k} , that capitulate in \mathbb{K}_j , for all $j \in \{1, 2, 3\}$. For this we need the following theorem.

Theorem 5.1 ([12]). *Let K/k be a cyclic extension of prime degree, then the number of classes that capitulate in K/k is: $[K : k][E_k : N_{K/k}(E_K)]$, where E_k and E_K are the unit groups of k and K respectively.*

Theorem 5.2. *Let \mathbb{K}_j , $1 \leq j \leq 3$, be the three unramified quadratic extensions of \mathbb{k} defined above.*

1. Let $\epsilon_{p_2q} = a + b\sqrt{p_2q}$.
 - i. If $x \pm 1$ is a square in \mathbb{N} and $a + 1$, $a - 1$ are not, then $|\kappa_{\mathbb{K}_1}| = 8$.
 - ii. If $a \pm 1$ and $(2p_1(x \pm 1)$ or $p_2(x \pm 1))$ are squares in \mathbb{N} , then $|\kappa_{\mathbb{K}_1}| = 2$.
 - iii. For the other cases $|\kappa_{\mathbb{K}_1}| = 4$.
2. Let $\epsilon_{p_1q} = a + b\sqrt{p_1q}$.
 - i. If $x \pm 1$ is a square in \mathbb{N} and $a + 1$, $a - 1$ are not, then $|\kappa_{\mathbb{K}_2}| = 8$.
 - ii. If $a \pm 1$ and $(2p_1(x \pm 1)$ or $p_2(x \pm 1))$ are squares in \mathbb{N} , then $|\kappa_{\mathbb{K}_2}| = 2$.
 - iii. For the other cases $|\kappa_{\mathbb{K}_2}| = 4$.
3. Let $\epsilon_{p_1p_2} = a + b\sqrt{p_1p_2}$.
 - i. If $N(\epsilon_{p_1p_2}) = 1$, then
 - a. If $x \pm 1$ is a square in \mathbb{N} , then $|\kappa_{\mathbb{K}_3}| = 4$.
 - b. Else $|\kappa_{\mathbb{K}_3}| = 2$.
 - ii. If $N(\epsilon_{p_1p_2}) = -1$, then
 - a. If $q(x \pm 1)$ or $2q(x \pm 1)$ is a square in \mathbb{N} , then $|\kappa_{\mathbb{K}_3}| = 2$.
 - b. Else $|\kappa_{\mathbb{K}_3}| = 4$.

Proof. Note first that, by Lemma 2.5, $E_{\mathbb{k}} = \langle i, \sqrt{i\epsilon_{p_1p_2q}} \rangle$ if $x \pm 1$ is a square in \mathbb{N} , and $E_{\mathbb{k}} = \langle i, \epsilon_{p_1p_2q} \rangle$ otherwise.

1. i. According to Proposition 3.1, if $x \pm 1$ is a square in \mathbb{N} and $a + 1$, $a - 1$ are not, then $N_{\mathbb{K}_1/\mathbb{k}}(E_{\mathbb{K}_1}) = \langle i, \epsilon_{p_1p_2q} \rangle$. Hence Theorem 5.1 yields that $|\kappa_{\mathbb{K}_1}| = 8$.

ii. if $a \pm 1$ and $(2p_1(x \pm 1)$ or $p_2(x \pm 1))$ are squares in \mathbb{N} , then by Proposition 3.1(1. and 3.) we get $N_{\mathbb{K}_1/\mathbb{k}}(E_{\mathbb{K}_1}) = \langle -1, i\epsilon_{p_1p_2q} \rangle$ or $\langle i, \epsilon_{p_1p_2q} \rangle$. Hence Theorem 5.1 yields that $|\kappa_{\mathbb{K}_1}| = 2$.

iii. If $a \pm 1$ and $x \pm 1$ are squares in \mathbb{N} , then $E_{\mathbb{k}} = \langle i, \sqrt{i\epsilon_{p_1p_2q}} \rangle$ and $N_{\mathbb{K}_1/\mathbb{k}}(E_{\mathbb{K}_1}) = \langle i, \epsilon_{p_1p_2q} \rangle$. Hence Theorem 5.1 yields that $|\kappa_{\mathbb{K}_1}| = 4$.

For the other cases, we have $E_{\mathbb{k}} = \langle i, \epsilon_{p_1p_2q} \rangle$ and $N_{\mathbb{K}_1/\mathbb{k}}(E_{\mathbb{K}_1}) = \langle i, \epsilon_{p_1p_2q}^2 \rangle$, $\langle -1, \epsilon_{p_1p_2q} \rangle$ or $\langle -1, i\epsilon_{p_1p_2q} \rangle$. Hence Theorem 5.1 yields that $|\kappa_{\mathbb{K}_1}| = 4$.

The other assertions of the theorem are similarly proved. \square

5.1. Capitulation in \mathbb{K}_1 . We begin this subsection by the following result.

Proposition 5.3. *Let d be a square-free integer and $p \equiv 1 \pmod{4}$ a prime divisor of d . Put $k = \mathbb{Q}(\sqrt{d}, i)$ and $p = \pi\pi'$, where π and π' are in $\mathbb{Q}(i)$. Let \mathcal{H} be a prime ideal of k above π , then \mathcal{H} capitulates in $K = k(\sqrt{p})$.*

Proof. It is easy to see that \mathcal{H} ramifies in $k/\mathbb{Q}(i)$ and it is of order 2. As $\epsilon_p = \frac{1}{2}(x + y\sqrt{p})$ it is of norm -1 , so $x^2 + 4 = y^2p$, hence by the decomposition uniqueness there exist y_1, y_2 in $\mathbb{Z}[i]$ such that

$$(1) \begin{cases} x \pm 2i &= y_1^2\pi \\ x \mp 2i &= y_2^2\pi', \end{cases} \quad \text{or} \quad (2) \begin{cases} x \pm 2i &= iy_1^2\pi \\ x \mp 2i &= -iy_2^2\pi'. \end{cases} \quad \text{with } p = \pi\pi', \ y = y_1y_2.$$

The system (1) implies that $2x = y_1^2\pi + y_2^2\pi'$. Put $\alpha = \frac{1}{2}(y_1\pi + y_2\sqrt{p})$ and $\beta = \frac{1}{2}(y_2\pi' + y_1\sqrt{p})$. Then α and β are in $K = k(\sqrt{p})$, and we have:

$$\begin{aligned}\alpha^2 &= \frac{1}{4}(y_1^2\pi^2 + y_2^2p + 2y_1y_2\pi\sqrt{p}) \\ &= \frac{1}{4}\pi(y_1^2\pi + y_2^2\pi' + 2y\sqrt{p}), \text{ since } p = \pi\pi' \text{ and } y = y_1y_2. \\ &= \frac{1}{4}\pi(2x + 2y\sqrt{p}), \text{ since } 2x = y_1^2\pi + y_2^2\pi'. \\ &= \pi\epsilon_p, \text{ since } \epsilon_p = \frac{1}{2}(x + y\sqrt{p}).\end{aligned}$$

And as ϵ_p is a unit of K , so the ideal generated by α^2 is equal to (π) . Thus $(\alpha^2) = (\pi) = \mathcal{H}^2$, hence $(\alpha) = \mathcal{H}$ and the result derived.

Similarly, the system (2) implies that $2x = iy_1^2\pi_2 - iy_2^2\pi_1$, thus $\alpha = \sqrt{\pi\epsilon_p} = \frac{1}{2}(y_1(1+i)\pi + y_2(1-i)\sqrt{p}) \in K$. Hence $\pi\epsilon_p = \alpha^2$ and $(\alpha) = \mathcal{H}$, so the result. \square

Theorem 5.4. *Keep the notations and hypotheses previously mentioned and put $\epsilon_{p_2q} = a + b\sqrt{p_2q}$, $\epsilon_{p_1p_2q} = x + y\sqrt{p_1p_2q}$.*

1. *If $x \pm 1$ is a square in \mathbb{N} and $a+1, a-1$ are not, then $\kappa_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_3\mathcal{H}_4] \rangle$.*
2. *If $a \pm 1$ and $(p_1(x \pm 1)$ or $2p_1(x \pm 1))$ are squares in \mathbb{N} , then $\kappa_{\mathbb{K}_1} = \langle [\mathcal{H}_1] \rangle$.*
3. *If $a + 1, a - 1$ are not squares in \mathbb{N} and $p_1(x \pm 1)$ or $2p_1(x \pm 1)$ is, then $\kappa_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{H}_3\mathcal{H}_4] \rangle$.*
4. *In the other cases we have: $\kappa_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$.*

Proof. By Proposition 5.3, \mathcal{H}_1 and \mathcal{H}_2 capitulate in \mathbb{K}_1 .

1. As $a + 1$ and $a - 1$ are not squares in \mathbb{N} , so, from the proof of Proposition 3.1, we get $p_2\epsilon_{p_2q}$ or $2p_2\epsilon_{p_2q}$ is a square in \mathbb{K}_1 . Therefore there exist $\alpha \in \mathbb{K}_1$ such that

$$(\alpha^2) = (p_2) = (\pi_2\pi_3) = (\mathcal{H}_3\mathcal{H}_4)^2 \text{ or } \left(\frac{\alpha}{1+i} \right)^2 = (p_2) = (\pi_2\pi_3) = (\mathcal{H}_3\mathcal{H}_4)^2.$$

Which implies that $\mathcal{H}_3\mathcal{H}_4$ capitulates in \mathbb{K}_1 . On the other hand, since $x \pm 1$ is a square in \mathbb{N} and $a + 1, a - 1$ are not, so proceeding as in Lemma 4.1, we prove that $\mathcal{H}_1\mathcal{H}_2$, $\mathcal{H}_1\mathcal{H}_3\mathcal{H}_4$, $\mathcal{H}_2\mathcal{H}_3\mathcal{H}_4$ and $\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\mathcal{H}_4$ are not principal in \mathbb{k} . Hence Theorem 5.2 implies the result.

2. Since $p_1(x \pm 1)$ or $2p_1(x \pm 1)$ are squares in \mathbb{N} and $(\mathcal{H}_1\mathcal{H}_2)^2 = (p_1)$, so, by [4, Proposition 2], $[\mathcal{H}_1] = [\mathcal{H}_2]$. Hence Theorem 5.2 implies the result.
3. Since $[\mathcal{H}_1] = [\mathcal{H}_2]$, the result is obvious by Theorem 5.2.
4. As $p_1(x + 1), p_1(x - 1), 2p_1(x + 1)$ and $2p_1(x - 1)$ are not squares in \mathbb{N} , so $[\mathcal{H}_1] \neq [\mathcal{H}_2]$. Hence Theorem 5.2 implies the result. \square

5.2. Capitulation in \mathbb{K}_2 . Since p_1 and p_2 play symmetric roles, so the capitulation of the 2-ideal classes of \mathbb{k} in $\mathbb{K}_2 = \mathbb{k}(\sqrt{p_2})$ is deduced from the previous Theorem 5.4.

Theorem 5.5. *Keep the notations and hypotheses previously mentioned and put $\epsilon_{p_1q} = a + b\sqrt{p_1q}$.*

1. *If $x \pm 1$ is a square in \mathbb{N} and $a+1, a-1$ are not, then $\kappa_{\mathbb{K}_2} = \langle [\mathcal{H}_3], [\mathcal{H}_4], [\mathcal{H}_1\mathcal{H}_2] \rangle$.*
2. *If $a \pm 1$ and $(p_2(x \pm 1) \text{ or } 2p_2(x \pm 1))$ are squares in \mathbb{N} , then $\kappa_{\mathbb{K}_2} = \langle [\mathcal{H}_3] \rangle$.*
3. *If $a+1$ and $a-1$ are not squares in \mathbb{N} and $p_2(x \pm 1) \text{ or } 2p_2(x \pm 1)$ is, then $\kappa_{\mathbb{K}_2} = \langle [\mathcal{H}_3], [\mathcal{H}_1\mathcal{H}_2] \rangle$.*
4. *In the other cases, $\kappa_{\mathbb{K}_2} = \langle [\mathcal{H}_3], [\mathcal{H}_4] \rangle$.*

5.3. Capitulation in \mathbb{K}_3 . Finally, we study the capitulation of the 2-ideal classes of \mathbb{k} in $\mathbb{K}_3 = \mathbb{k}(\sqrt{q}) = \mathbb{k}(\sqrt{p_1p_2})$.

Theorem 5.6. *Keep the notations and hypotheses previously mentioned and assume $N(\epsilon_{p_1p_2}) = 1$.*

1. *If $x \pm 1$ is a square in \mathbb{N} , then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_1\mathcal{H}_2], [\mathcal{H}_3\mathcal{H}_4] \rangle$.*
2. *If $p_1(x \pm 1) \text{ or } 2p_1(x \pm 1)$ is a square in \mathbb{N} , then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_3\mathcal{H}_4] \rangle$.*
3. *If $p_2(x \pm 1) \text{ or } 2p_2(x \pm 1)$ is a square in \mathbb{N} , then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_1\mathcal{H}_2] \rangle$.*
4. *If $q(x \pm 1) \text{ or } 2q(x \pm 1)$ is a square in \mathbb{N} , then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_1\mathcal{H}_2] \rangle = \langle [\mathcal{H}_3\mathcal{H}_4] \rangle$.*

Proof. According to the previous theorems $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ and \mathcal{H}_4 are not principal in \mathbb{k} . On the other hand, from the proof of Proposition 3.3 we know that $2p_1\epsilon_{p_1p_2}$ and $2p_2\epsilon_{p_1p_2}$ or $p_1\epsilon_{p_1p_2}$ and $p_2\epsilon_{p_1p_2}$ are squares in \mathbb{K}_3 . Thus there exist α, β in \mathbb{K}_3 such that $\mathcal{H}_1\mathcal{H}_2 = (\frac{\alpha}{1+i})$ and $\mathcal{H}_3\mathcal{H}_4 = (\frac{\beta}{1+i})$ or $\mathcal{H}_1\mathcal{H}_2 = (\alpha)$ and $\mathcal{H}_3\mathcal{H}_4 = (\beta)$, which implies that $\mathcal{H}_1\mathcal{H}_2$ and $\mathcal{H}_3\mathcal{H}_4$ capitulate in \mathbb{K}_3 . We have four cases to distinguish.

1. Suppose $x \pm 1$ is a square in \mathbb{N} . Since $(\mathcal{H}_1\mathcal{H}_2)^2 = (p_1)$, $(\mathcal{H}_3\mathcal{H}_4)^2 = (p_2)$ and $(\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\mathcal{H}_4)^2 = (p_1p_2)$, then Proposition 2 and Remark 1 of [4] yield that $\mathcal{H}_1\mathcal{H}_2, \mathcal{H}_3\mathcal{H}_4$ and $\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\mathcal{H}_4$ are not principal in \mathbb{k} ; hence $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_1\mathcal{H}_2], [\mathcal{H}_3\mathcal{H}_4] \rangle$.
2. If $p_1(x \pm 1) \text{ or } 2p_1(x \pm 1)$ is a square in \mathbb{N} , then [4, Proposition 2] yields that $[\mathcal{H}_1\mathcal{H}_2] = 1$; so the result.
3. If $p_2(x \pm 1) \text{ or } 2p_2(x \pm 1)$ is a square in \mathbb{N} , then [4, Proposition 2] yields that $[\mathcal{H}_3\mathcal{H}_4] = 1$; so the result.
4. If $q(x \pm 1) \text{ or } 2q(x \pm 1)$ is a square in \mathbb{N} , then [4, Remark 1] yields that $\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\mathcal{H}_4$ is principal in \mathbb{k} , so the result.

□

Theorem 5.7. *Keep the notations and hypotheses previously mentioned and assume $N(\epsilon_{p_1p_2}) = -1$.*

1. *If $q(x \pm 1) \text{ or } 2q(x \pm 1)$ is a square in \mathbb{N} , then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_1\mathcal{H}_3] \rangle \text{ or } \langle [\mathcal{H}_1\mathcal{H}_4] \rangle$.*
2. *If $x \pm 1$ is a square in \mathbb{N} , then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_1\mathcal{H}_3], [\mathcal{H}_2\mathcal{H}_4] \rangle \text{ or } \langle [\mathcal{H}_1\mathcal{H}_4], [\mathcal{H}_2\mathcal{H}_3] \rangle$.*
3. *If $p_1(x \pm 1) \text{ or } 2p_1(x \pm 1)$ is a square in \mathbb{N} , then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_1\mathcal{H}_3], [\mathcal{H}_1\mathcal{H}_4] \rangle$.*
4. *If $p_2(x \pm 1) \text{ or } 2p_2(x \pm 1)$ is a square in \mathbb{N} , then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_1\mathcal{H}_3], [\mathcal{H}_2\mathcal{H}_3] \rangle$.*

Proof. As $N(\epsilon_{p_1 p_2}) = -1$, then by the decomposition uniqueness in $\mathbb{Z}[i]$, there exist b_1 and b_2 in $\mathbb{Z}[i]$ such that

$$\begin{cases} a \pm i = b_1^2 \pi_1 \pi_3, \\ a \mp i = b_2^2 \pi_2 \pi_4, \end{cases} \quad \text{or} \quad \begin{cases} a \pm i = ib_1^2 \pi_1 \pi_3, \\ a \mp i = -ib_2^2 \pi_2 \pi_4, \end{cases} \quad \text{or} \\ \begin{cases} a \pm i = b_1^2 \pi_1 \pi_4, \\ a \mp i = b_2^2 \pi_2 \pi_3, \end{cases} \quad \text{or} \quad \begin{cases} a \pm i = ib_1^2 \pi_1 \pi_4, \\ a \mp i = -ib_2^2 \pi_2 \pi_3, \end{cases}$$

with $p_1 = \pi_1 \pi_2$, $p_2 = \pi_3 \pi_4$, $y = y_1 y_2$ and π_2 (resp. π_4 , y_2) is the complex conjugate of π_1 (resp. π_3 , y_1). Therefore $\sqrt{2\epsilon_{p_1 p_2}} = b_1 \sqrt{\pi_1 \pi_3} + b_2 \sqrt{\pi_2 \pi_4}$ or $\sqrt{\epsilon_{p_1 p_2}} = b_1(1+i)\sqrt{\pi_1 \pi_3} + b_2(1-i)\sqrt{\pi_2 \pi_4}$ or $\sqrt{2\epsilon_{p_1 p_2}} = b_1 \sqrt{\pi_1 \pi_4} + b_2 \sqrt{\pi_2 \pi_3}$ or $\sqrt{\epsilon_{p_1 p_2}} = b_1(1+i)\sqrt{\pi_1 \pi_4} + b_2(1-i)\sqrt{\pi_2 \pi_3}$, hence $2\pi_1 \pi_3 \epsilon_{p_1 p_2}$ and $2\pi_2 \pi_4 \epsilon_{p_1 p_2}$ or $\pi_1 \pi_3 \epsilon_{p_1 p_2}$ and $\pi_2 \pi_4 \epsilon_{p_1 p_2}$ or $2\pi_1 \pi_4 \epsilon_{p_1 p_2}$ and $2\pi_2 \pi_3 \epsilon_{p_1 p_2}$ or $\pi_1 \pi_4 \epsilon_{p_1 p_2}$ and $\pi_2 \pi_3 \epsilon_{p_1 p_2}$ are squares in \mathbb{K}_3 . Thus there exist α, β in \mathbb{K}_3 such that $(\alpha^2) = (2\pi_1 \pi_3)$ and $(\beta^2) = (2\pi_2 \pi_4)$ or $(\alpha^2) = (\pi_1 \pi_3)$ and $(\beta^2) = (\pi_2 \pi_4)$ or $(\alpha^2) = (2\pi_1 \pi_4)$ and $(\beta^2) = (2\pi_2 \pi_3)$ or $(\alpha^2) = (\pi_1 \pi_4)$ and $(\beta^2) = (\pi_2 \pi_3)$. Consequently $\mathcal{H}_1 \mathcal{H}_3 = (\frac{\alpha}{1+i})$ and $\mathcal{H}_2 \mathcal{H}_4 = (\frac{\beta}{1+i})$ or $\mathcal{H}_1 \mathcal{H}_3 = (\alpha)$ and $\mathcal{H}_2 \mathcal{H}_4 = (\beta)$ or $\mathcal{H}_1 \mathcal{H}_4 = (\frac{\alpha}{1+i})$ and $\mathcal{H}_2 \mathcal{H}_3 = (\frac{\beta}{1+i})$ or $\mathcal{H}_1 \mathcal{H}_4 = (\alpha)$ and $\mathcal{H}_2 \mathcal{H}_3 = (\beta)$. Thus $\mathcal{H}_1 \mathcal{H}_3$ and $\mathcal{H}_2 \mathcal{H}_4$ or $\mathcal{H}_1 \mathcal{H}_4$ and $\mathcal{H}_2 \mathcal{H}_3$ capitulate in \mathbb{K}_3 . On the other hand, $\mathcal{H}_1 \mathcal{H}_3$ is not principal in \mathbb{k} , since $(\mathcal{H}_1 \mathcal{H}_3)^2 = (\pi_1 \pi_3) = (m + in)$ and $\sqrt{m^2 + n^2} = \sqrt{p_1 p_2} \notin \mathbb{k}$. Thus [4, Proposition 1] guarantees the result. We similarly show that $\mathcal{H}_1 \mathcal{H}_4$, $\mathcal{H}_2 \mathcal{H}_4$ and $\mathcal{H}_2 \mathcal{H}_3$ are not principal in \mathbb{k} .

1. If $q(x \pm 1)$ or $2q(x \pm 1)$ is a square in \mathbb{N} , then $p_1 p_2(x \pm 1)$ or $2p_1 p_2(x \pm 1)$ is a square in \mathbb{N} ; so [4, Remark 1] yields that $\mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3 \mathcal{H}_4$ is principal in \mathbb{k} , thus $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_1 \mathcal{H}_3] \rangle$ or $\langle [\mathcal{H}_1 \mathcal{H}_4] \rangle$.

2., 3. and 4. are similarly shown. \square

From Theorems 5.4, 5.5, 5.6, 5.7 and from Proposition 4.3, we deduce the following result.

Theorem 5.8. *Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod{4}$ be different primes. Put $\mathbb{k} = \mathbb{Q}(\sqrt{p_1 p_2 q}, i)$ and denote by $\mathbb{k}^{(*)}$ its genus field. Let $\epsilon_{p_1 p_2 q} = x + y\sqrt{p_1 p_2 q}$ be the fundamental unit of $\mathbb{Q}(\sqrt{p_1 p_2 q})$.*

1. *If $x \pm 1$ is a square in \mathbb{N} , then $\langle \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4 \rangle \subseteq \kappa_{\mathbb{k}^{(*)}}$.*
2. *If $p_1(x \pm 1)$ or $2p_1(x \pm 1)$ is a square in \mathbb{N} , then $\langle \mathcal{H}_1, \mathcal{H}_3, \mathcal{H}_4 \rangle \subseteq \kappa_{\mathbb{k}^{(*)}}$.*
3. *If $p_2(x \pm 1)$ or $2p_2(x \pm 1)$ is a square in \mathbb{N} , then $\langle \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3 \rangle \subseteq \kappa_{\mathbb{k}^{(*)}}$.*
4. *If $q(x \pm 1)$ or $2q(x \pm 1)$ is a square in \mathbb{N} , then $\langle \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3 \rangle = \langle \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_4 \rangle \subseteq \kappa_{\mathbb{k}^{(*)}}$.*

Theorem 5.8 implies the following corollary:

Corollary 5.9. *Let $\mathbb{k} = \mathbb{Q}(\sqrt{p_1 p_2 q}, i)$, where $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod{4}$ are different primes. Let $\mathbb{k}^{(*)}$ be the genus field of \mathbb{k} and $\text{Am}_s(\mathbb{k}/\mathbb{Q}(i))$ be the group of the strongly ambiguous class of $\mathbb{k}/\mathbb{Q}(i)$, then $\text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) \subseteq \kappa_{\mathbb{k}^{(*)}}$.*

6. Application

Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod{4}$ be different primes such that $\mathbf{Cl}_2(\mathbb{k})$ is of type $(2, 2, 2)$. According to [3], $\mathbf{Cl}_2(\mathbb{k})$ is of type $(2, 2, 2)$ if and only if p_1 , p_2 and q satisfy the following conditions:

- $p_1 \equiv 5$ or $p_2 \equiv 5 \pmod{8}$.
- Two at least of the elements of $\left\{ \left(\frac{p_1}{p_2} \right), \left(\frac{p_1}{q} \right), \left(\frac{p_2}{q} \right) \right\}$ are equal to -1 .

These conditions are detailed in three types *I*, *II* and *III* and each type consists of three cases (a), (b) and (c) (see [4]). To continue we need the following results.

Lemma 6.1. *Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod{4}$ be different primes as above.*

1. *If p_1 , p_2 and q are of type I, then $p_1(x \pm 1)$ is a square in \mathbb{N} .*
2. *If p_1 , p_2 and q are of type II, then $p_2(x \pm 1)$ is a square in \mathbb{N} .*
3. *If p_1 , p_2 and q are of type III, then $q(x \pm 1)$ i.e. $p_1 p_2(x \mp 1)$ is a square in \mathbb{N} .*

Proof. As $p_1 \equiv 5$ or $p_2 \equiv 5 \pmod{8}$, so the unit index of \mathbb{k} is 1 (see [3, Corollaire 3.2]). On the other hand, $N(\varepsilon_d) = 1$ i.e. $x^2 - 1 = y^2 p_1 p_2 q$, hence by Lemma 2.5, $x \pm 1$ is not a square in \mathbb{N} . Thus by the decomposition uniqueness in \mathbb{Z} and by Lemma 2.2, there exist y_1, y_2 in \mathbb{Z} such that:

$$(1) \begin{cases} x \pm 1 = p_1 y_1^2, \\ x \mp 1 = p_2 q y_2^2; \end{cases} \text{ or } (2) \begin{cases} x \pm 1 = 2p_1 y_1^2, \\ x \mp 1 = 2p_2 q y_2^2; \end{cases} \text{ or } (3) \begin{cases} x \pm 1 = p_2 y_1^2, \\ x \mp 1 = p_1 q y_2^2; \end{cases} \text{ or } (4) \begin{cases} x \pm 1 = 2p_2 y_1^2, \\ x \mp 1 = 2p_1 q y_2^2; \end{cases} \text{ or } (5) \begin{cases} x \pm 1 = q y_1^2, \\ x \mp 1 = p_1 p_2 y_2^2; \end{cases} \text{ or } (6) \begin{cases} x \pm 1 = 2q y_1^2, \\ x \mp 1 = 2p_1 p_2 y_2^2; \end{cases}$$

1. Suppose p_1 , p_2 and q are of type I, then this contradicts systems (2), (3), (4), (5) and (6), since:

- system (2) implies that $\left(\frac{p_1}{p_2} \right) = \left(\frac{p_1}{q} \right) = 1$
- system (3) implies that $\left(\frac{p_1}{p_2} \right) = \left(\frac{2}{p_1} \right)$ and $\left(\frac{p_1 q}{p_2} \right) = \left(\frac{2}{p_2} \right)$
- system (4) implies that $\left(\frac{p_1}{p_2} \right) = \left(\frac{p_2}{q} \right) = 1$
- system (5) implies that $\left(\frac{p_1}{q} \right) = \left(\frac{2}{p_1} \right) = 1$
- system (6) implies that $\left(\frac{p_1}{q} \right) = 1$.

Thus only the system (1) occurs, which yields that $p_1(x \pm 1)$ is a square in \mathbb{N} and $p_2(x \pm 1)$, $2p_2(x \pm 1)$ are not. □

2. and 3. are similarly checked.

Proceeding similarly, we prove the following lemma.

Lemma 6.2. *Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod{4}$ be different primes and put $\varepsilon_{p_2 q} = a + b\sqrt{p_2 q}$.*

1. *If p_1 , p_2 and q are of type I(a), then $p_2(a \pm 1)$ or $2p_2(a \pm 1)$ is a square in \mathbb{N} .*
2. *If p_1 , p_2 and q are of type I(b), then $p_2(a \pm 1)$ is a square in \mathbb{N} .*

3. If p_1, p_2 and q are of type $I(c)$ or $II(a)$, then $a \pm 1$ is a square in \mathbb{N} .
4. If p_1, p_2 and q are of type $II(c)$ or $III(a)$ or $III(b)$, then $p_2(a \pm 1)$ is a square in \mathbb{N} .
5. If p_1, p_2 and q are of type $II(b)$ or $III(c)$, then $2p_2(a \pm 1)$ is a square in \mathbb{N} .

Denote by \mathcal{H}_1 and \mathcal{H}_2 (resp. \mathcal{H}_3 and \mathcal{H}_4) the prime ideals of \mathbb{k} above p_1 (resp. p_2), then we have:

Lemma 6.3 ([4]). *Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod{4}$ be different primes and assume $\mathbf{Cl}_2(\mathbb{k}) \simeq (2, 2, 2)$.*

1. *If p_1, p_2 and q are of type I , then $\mathbf{Cl}_2(\mathbb{k}) = \langle [\mathcal{H}_1], [\mathcal{H}_3], [\mathcal{H}_4] \rangle$.*
2. *If p_1, p_2 and q are of type II or III , then $\mathbf{Cl}_2(\mathbb{k}) = \langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_3] \rangle$.*

Remark 6.4. If $\mathbf{Cl}_2(\mathbb{k})$ is of type $(2, 2, 2)$, then by Proposition 4.3 and Lemma 6.3, we deduce that $\text{Am}(\mathbb{k}/\mathbb{Q}(i)) = \text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \mathbf{Cl}_2(\mathbb{k})$.

Theorem 6.5. *Let $\mathbb{k} = \mathbb{Q}(\sqrt{d}, i)$, where $d = p_1 p_2 q$ with p_1, p_2 and q are different primes such that $\mathbf{Cl}_2(\mathbb{k})$, the 2-class groupe of \mathbb{k} , is of type $(2, 2, 2)$.*

1. *If p_1, p_2 and q are of type $I(c)$, then $\kappa_{\mathbb{K}_1} = \langle [\mathcal{H}_1] \rangle$.*
2. *If p_1, p_2 and q are of type $I(a)$ or $I(b)$, then $\kappa_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{H}_3 \mathcal{H}_4] \rangle$.*
3. *If p_1, p_2 and q are of type II or III , then $\kappa_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$.*

Proof. From Lemmas 6.1, 6.2 and 6.3 we get:

1. If p_1, p_2 and q are of type $I(c)$, then $a \pm 1$ and $p_1(x \pm 1)$ are squares in \mathbb{N} . Hence Theorem 5.4 implies the result.
2. If p_1, p_2 and q are of type $I(a)$ or $I(b)$, then $p_2(a \pm 1)$ or $2p_2(a \pm 1)$ is a square in \mathbb{N} , and since $p_1(x \pm 1)$ is a square in \mathbb{N} , hence Theorem 5.4 implies the result.
3. a. If p_1, p_2 and q are of type II , then $a \pm 1$ or $p_2(a \pm 1)$ or $2p_2(a \pm 1)$ is a square in \mathbb{N} , and as in this case $p_2(x \pm 1)$ is also a square in \mathbb{N} , hence Theorem 5.4 implies the result.
- b. If p_1, p_2 and q are of type III , then $p_2(a \pm 1)$ or $2p_2(a \pm 1)$ is a square in \mathbb{N} , and since $q(x \pm 1)$ is also a square in \mathbb{N} , hence Theorem 5.4 implies the result.

□

As p_1 and p_2 play symmetric roles, so with a similar argument to that used in the previous theorem, we deduce the following theorem. Note that in this case \mathcal{H}_3 and \mathcal{H}_4 always capitulate in \mathbb{K}_2 (Proposition 5.3). Note also that whenever p_1, p_2 and q are of type II , then $[\mathcal{H}_3] = [\mathcal{H}_4]$ since in this case $p_2(x \pm 1)$ is a square in \mathbb{N} and the result is guaranteed by [4, Proposition 1]. Finally, note that if p_1, p_2 and q are of type III , then \mathcal{Q} , the prime ideal of \mathbb{k} lies above q , is principal in \mathbb{k} ; hence $\mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3 \mathcal{H}_4$ is too.

Theorem 6.6. *Keep the hypotheses and notations mentioned in Theorem 6.5.*

1. *If p_1, p_2 and q are of type $II(c)$, then $\kappa_{\mathbb{K}_2} = \langle [\mathcal{H}_3] \rangle$.*

2. If p_1, p_2 and q are of type $II(a)$ or $II(b)$ or III , then $\kappa_{\mathbb{K}_2} = \langle [\mathcal{H}_3], [\mathcal{H}_1\mathcal{H}_2] \rangle$.
3. If p_1, p_2 and q are of type I , then $\kappa_{\mathbb{K}_2} = \langle [\mathcal{H}_3], [\mathcal{H}_4] \rangle$.

Finally, we compute the 2-idea classes of \mathbb{k} that capitulate in $\mathbb{K}_3 = \mathbb{Q}(\sqrt{q}, \sqrt{p_1p_2}, i)$.

Theorem 6.7. *Keep the hypotheses and notations mentioned in Theorem 6.5 and assume $N(\varepsilon_{p_1p_2}) = 1$.*

1. If p_1, p_2 and q are of type I , then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_3\mathcal{H}_4] \rangle$.
2. If p_1, p_2 and q are of type II or III , then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_1\mathcal{H}_2] \rangle$.

Proof. From Lemmas 6.1 and 6.3 we get:

1. If p_1, p_2 and q are of type $I(c)$, then then $p_1(x \pm 1)$ is a square in \mathbb{N} . Hence Theorem 5.6 implies the result.
2. a. If p_1, p_2 and q are of type II , then $p_2(x \pm 1)$ is a square in \mathbb{N} , hence Theorem 5.6 implies the result.
- b. If p_1, p_2 and q are of type III , then $q(x \pm 1)$ i.e. $p_1p_2(x \pm 1)$ is a square in \mathbb{N} , hence Theorem 5.6 implies the result.

□

Theorem 6.8. *Keep the hypotheses and notations mentioned in Theorem 6.5 and assume $N(\varepsilon_{p_1p_2}) = -1$.*

1. If p_1, p_2 and q are of type III , then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_1\mathcal{H}_3] \rangle$ or $\langle [\mathcal{H}_2\mathcal{H}_3] \rangle$.
2. If p_1, p_2 and q are of type II , then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_1\mathcal{H}_3], [\mathcal{H}_2\mathcal{H}_3] \rangle$.
3. If p_1, p_2 and q are of type I , then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_1\mathcal{H}_3], [\mathcal{H}_1\mathcal{H}_4] \rangle$.

Proof. It is a simple deduction from Theorem 5.7 and Lemma 6.1. □

From Theorems 6.5, 6.6, 6.7 and 6.8, we deduce the following corollary .

Corollary 6.9. *Keep the hypotheses and notations mentioned in Theorem 6.5. Then all the classes of $\mathbf{Cl}_2(\mathbb{k})$ capitulate in $\mathbb{k}^{(*)}$ i.e.*

$$\kappa_{\mathbb{k}^{(*)}} = \mathbf{Cl}_2(\mathbb{k}) = \text{Am}(\mathbb{k}/\mathbb{Q}(i)) = \text{Am}_s(\mathbb{k}/\mathbb{Q}(i)).$$

7. PROOF OF THE MAIN THEOREM

The main Theorem is a simple deduction from Proposition 4.3, Theorems 5.2, 5.4, 5.5, 5.6, 5.7 and Corollaries 5.9, 6.9.

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